Fibre Bundles

DEFINITION 1. A *fibre bundle* consists of 3 topological spaces F, E, and B together with a continous surjection $\pi : E \to B$. Additionally, we require there to exist a set $\{U_\alpha, \psi_\alpha\}$ (called a *local trivialization*) where the U_α are an open cover of B and each ψ_α is a homeomorphism, $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$, that makes the following diagram commute:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & U_\alpha \times F \\ \downarrow \pi & \swarrow proj_1 & \\ U_\alpha & & \end{array}$$

Note $\pi^{-1}(x) \cong F$ for each $x \in B$. $\pi^{-1}(x)$ is called the *fibre* of E over x. We will frequently write fibre bundles in the form $F \hookrightarrow E \xrightarrow{\pi} B$.

EXAMPLE 1. The *trivial bundle* over B with fibre F is the bundle $F \hookrightarrow B \times F \xrightarrow{\pi} B$ where π is the map that projects $B \times F$ to B.

If a fibre bundle $F \hookrightarrow E \xrightarrow{\pi} B$ admits a homeomorphism $\psi : \pi^{-1}(U) \to U \times F$ for some open set $U \subset B$, we say the bundle is *trivial* over U and that U is a *trivializing neighborhood* of E.

DEFINITION 2. Given a topologicial group G, a *principal G-bundle* is a fibre bundle $G \hookrightarrow P_G \to B$ together with a local trivialization (called a *G-atlas*) $\{U_\alpha, \psi_\alpha\}$ where for each $U_i \cap U_j \neq \emptyset$, the map $\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times G \to (U_i \cap U_j) \times G$ is given by $(x, g) \mapsto (x, t_{ij}(x)g)$ where t_{ij} is a continous map, $t_{ij} : U_i \cap U_j \to G$ (called a *transition function*).

LEMMA 1. *If $G \hookrightarrow P_G \xrightarrow{\pi} B$ is a principal G bundle then the right multiplication action of G on itself induces an action of G on P_G that preserves fibres and acts freely and transitively on them.*

PROOF. Let $(U_\alpha, \varphi_\alpha)$ be the given G-atlas for P_G. Given a $\varphi : \pi^{-1}(U) \to U \times G$, we have an action of G on $\pi^{-1}(U)$ given by $\ell \mapsto \varphi^{-1} \circ r_\ell \circ \varphi$ where r_ℓ denotes right multiplication of $\ell \in G$ on $U \times G$. Note this action preserves the fibres of $\pi^{-1}(U)$ and acts freely and transitively on them.

Now consider another $\psi : \pi^{-1}(U) \to U \times G$. Each map $\psi \circ \varphi^{-1} : \{x\} \times G \to \{x\} \times G$ is given by $(x, g) \mapsto (x, hg)$ for some $h \in G$. Since $(hg)\ell = h(g\ell)$, we have $r_\ell \circ \varphi \circ \psi^{-1} = \varphi \circ \psi^{-1} \circ r_\ell$. It follows $\varphi \circ r_\ell \circ \varphi^{-1} = \psi \circ r_\ell \circ \psi^{-1}$.

So we see our induced action is independent of the choice of φ. By considering all U_α, we obtain the desired action.

$\square$

Throughout this section, let K denote the field $\mathbb{R}$ (or $\mathbb{C}$).

DEFINITION 3. An n-dimensional real (complex) vector bundle is a fibre bundle $K^n \hookrightarrow E \xrightarrow{\pi} B$ together with an n-dimensional real (complex) vector space structure for each fibre, $\pi^{-1}(x)$. We require there to exist a local trivialization $\{U_\alpha, \psi_\alpha\}$ where the restriction of each $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times K^n$ to each $\pi^{-1}(x) \subset \pi^{-1}(U_\alpha)$ is a linear map, $\pi^{-1}(x) \to \{x\} \times K^n$.

REMARK 1. We will also define vector bundles as above only with given fibres that are topological vector spaces. We require each given fibre to admit a homeomorphism with K^n for some n that is also a vector space isomorphism, so the two definitions coincide.

Let G be a subgroup of $GL(n, K)$. A G-*structure* for a vector bundle $K^n \hookrightarrow E \to B$ is a local trivialization $\{U_\alpha, \psi_\alpha\}$ where for each $U_i \cap U_j \neq \emptyset$, the map $\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times K^n \to (U_i \cap U_j) \times K^n$ is given by $(x, v) \mapsto (x, t_{ij}(x)v)$ where t_{ij} is a continous map, $t_{ij} : U_i \cap U_j \to G$. We say a vector bundle endowed with a G-structure is a vector bundle with structure group G.

A *section* of a vector bundle $K^n \hookrightarrow E \xrightarrow{\pi} B$ is a continous map $s : B \to E$ with $\pi \circ s = id_B$. The set of sections of E has an infinite dimensional real (complex) vector space structure where addition and scalar multiplication are defined fibrewise. We denote this vector space by $\Gamma(E)$.

DEFINITION 4. Given two fibre bundle $F_1 \hookrightarrow E_1 \xrightarrow{\pi_1} B$ and $F_2 \hookrightarrow E_2 \xrightarrow{\pi_2} B$, a bundle map is a continous map $\varphi : E_1 \to E_2$ where the following commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ \downarrow{\scriptstyle \pi_1} & \swarrow{\scriptstyle \pi_2} & \\ B & & \end{array}$$

- If E_1 and E_2 are both vector bundles, we require each restriction $\varphi : \pi_1^{-1}(x) \to \pi_2^{-1}(x)$ to be a linear map.
- If E_1 and E_2 are both principal bundles with fibres G and H respectively, then we require there to exist a group homomorphism $\phi : G \to H$ satisfying $\varphi(p \cdot g) = \varphi(p) \cdot \phi(g)$ for each $p \in P_G$ and $g \in G$. We say the bundle map is ϕ fibrewise.
- If φ is a homeomorphism, we say it is a *fibre bundle isomorphism*.

REMARK 2. Bundle maps are also defined between fibre bundles with distinct base spaces given a continous map between the base spaces. However we will not use this generalization here.

Let $K^n \hookrightarrow E_1 \to B$ and $K^m \hookrightarrow E_2 \to B$ denote two vector bundles. Since each $x \in E_1$ is contained in the image of some section $s : B \to E_1$, a bundle map $\varphi : E_1 \to E_2$ induces a linear map $\varphi : \Gamma(E_1) \to \Gamma(E_2)$. On the other hand, we will only consider linear maps $\varphi : \Gamma(E_1) \to \Gamma(E_2)$ that induce bundle maps.

DEFINITION 5. Given two fibre bundles $F_1 \hookrightarrow E_1 \xrightarrow{\pi_1} B$ and $F_2 \hookrightarrow E_2 \xrightarrow{\pi_2} B$, their *direct sum bundle* is the bundle $F_1 \oplus F_2 \hookrightarrow E_1 \oplus E_2 \xrightarrow{\pi} B$ where $\pi : E_1 \times E_2 \to B$ is the map $\pi_1 \oplus \pi_2$.

Note the direct sum bundle of two vector bundles is another vector bundle and the direct sum bundle of two principal bundles is another principal bundle.

Next we will define the *tensor product bundle* of two vector bundles, $K^n \hookrightarrow E_1 \xrightarrow{\pi_1} B$ and $K^m \hookrightarrow E_1 \xrightarrow{\pi_2} B$. First let $E_1 \otimes E_2$ denote the disjoint union of all the vector spaces, $\pi_1^{-1}(x) \otimes \pi_2^{-1}(x)$. We can define a projection map $\pi : E_1 \otimes E_2 \to B$ by sending each $\pi_1^{-1}(x) \otimes \pi_2^{-1}(x) \subset E_1 \otimes E_2$ to $x \in B$. We will define a topology on $E_1 \otimes E_2$ as follows. For each open set $U \subset B$ over which both E_1 and E_2 are trivial, choose two homeomorphisms, $\psi_1 : \pi_1^{-1}(U) \to U \times K^n$ and $\psi_2 : \pi_2^{-1}(U) \to U \times K^m$. We can define a map $\psi_1 \otimes \psi_2 : \pi^{-1}(U) \to U \times (K^n \otimes K^m)$ whose restriction to each $\pi_1^{-1}(x) \otimes \pi_2^{-1}(x) \subset \pi^{-1}(U)$ is the linear map $(v \otimes w) \mapsto \{x\} \times (\psi_1(v) \otimes \psi_2(w))$. There is a unique topology on $E_1 \otimes E_2$ where each of the $\psi_1 \otimes \psi_2$ maps is a homeomorphism and this topology doesn't depend on the choice of ψ_1 and ψ_2 for each U. If we endow $E_1 \otimes E_2$ with this topology, we obtain a real (complex) vector bundle $K^n \otimes K^m \hookrightarrow E_1 \otimes E_2 \xrightarrow{\pi} B$.

REMARK 3. We will denote the tensor product of a vector bundle $K^n \hookrightarrow E \to B$ with the trivial vector bundle with fibre K^m as $E \otimes K^m$. Similarly for a topological space F_2, we denote the direct sum of a fibre bundle $F_1 \hookrightarrow E \to B$ with the trivial bundle with fibre F_2 as $E \oplus F_2$.

Given a vector bundle $K^n \hookrightarrow E \xrightarrow{\pi} B$ with structure group $G \subset GL(n, K)$, we define the *frame bundle* of E as follows (recall a frame is simply an ordered basis). Let $P_G = \bigsqcup_\alpha U_\alpha \times G/\sim$ where the U_α are the open cover from the given G-structure. $\sim$ is the equivalence relation defined by $(x, g) \sim (x, h)$ whenever $t_{ij}(x)g = h$ for some transition function $t_{ij} : U_i \cap U_j \to G$ from the given G-structure. $G \hookrightarrow P_G \xrightarrow{\pi'} B$ will be a principal G-bundle. Here π' denotes the natural map that sends each $[(x, g)] \in P_G$ to $x \in B$.

Now we will define the *associated bundle construction*. Let $\pi : P_G \to B$ be a principal G-bundle over B and $\phi : G \to GL(n, K)$ be a homomorphism. Note G acts on $P_G \times K^n$ via $\varphi_g(p, v) = (pg^{-1}, \phi(g)v)$.

$$P_G \times_\phi K^n = P_G \times K^n/\varphi$$

will be an n-dimensional real (complex) vector bundle with structure group $\mathrm{Im}(\phi)$ and projection map $\pi'([p, v]) = \pi(p)$.

REMARK 4. When the homomorphism $\phi : G \to GL(n, K)$ is clear, we will write $P_G \times_G K^n$.

Note if $K^n \hookrightarrow E \xrightarrow{\pi} B$ is a n-dimensional (complex) vector bundle with structure group $G \subset GL(n, K)$ and frame bundle P_G then E is canonically isomorphic to the bundle $P_G \times_G K^n$ via the map $[p, v] \mapsto p(v)$. Recall a frame p of $\pi^{-1}(x)$ can be considered as the isomorphism $p : K^n \to \pi^{-1}(x)$ that sends the standard frame of K^n to the frame of $\pi^{-1}(x)$ specified by p.